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2002 J. Phys. A: Math. Gen. 35 L653

(http://iopscience.iop.org/0305-4470/35/45/104)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L653–L659 PII: S0305-4470(02)38691-8

LETTER TO THE EDITOR

Elliptic discrete Painlevé equations

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Received 25 June 2002
Published 29 October 2002
Online at stacks.iop.org/JPhysA/35/L653

Abstract
We present explicit forms of discrete Painlevé equations in which the independent variable and the parameter enter through the arguments of elliptic functions. These equations have eight degrees of freedom and the geometry of their transformations is described by the affine Weyl group $E_8^{(1)}$.

PACS numbers: 02.30.Ik, 02.30.Ks

The geometrical description of discrete Painlevé (d-$\mathbb{P}$) equations in terms of affine Weyl groups has revealed one interesting (and unexpected) feature [1]. While all discrete equations described by Weyl groups with fewer parameters than $E_8^{(1)}$ are either difference-equations or q-equations, among those mappings described by $E_8^{(1)}$ there exists a third variety. For the latter, the independent variable and the parameters enter through the arguments of elliptic functions. These are the elliptic d-$\mathbb{P}$ the title of the paper is referring to. In [2] we have studied in detail the geometry of the weight lattice of the $E_8^{(1)}$ affine Weyl group. As in all our previous studies based on the so-called grand scheme approach [3] our main assumption was that the multidimensional τ-function lives on the vertices of this lattice. More precisely, these points are the ones the coordinates of which are either all integer or all half-integer with the additional constraint that the sum of all coordinates is even. One can check that any such point (say, the origin) has 240 nearest-neighbours (at distance $\sqrt{2}$) and 2160 next-nearest-neighbours (at distance 2). On such points one can define a dependent variable, namely the τ-function which turns out to be the discrete equivalent of an entire function: it is always well determined and finite. We will use the symbol τ indifferently for the point and the function defined there. The next step is to consider midpoints of two τ at distance 2 (next-nearest-neighbours). Any such point is in fact the midpoint of eight distinct pairs of τ at distance 2. On such a point X we will define a variable for the nonlinear (as opposed to bilinear) equation. We will also call this variable X. Let ϕ_j be the product of the two τ of one pair of points at distance 2, and C_j the scalar product of the vector joining these points with the vector $\overrightarrow{\Omega X}$, where Ω is some arbitrary fixed point (which may well be distinct from the origin). Since the former vector is defined only up to a sign, so is the quantity C_j. We assume that there exist two *even* (because C_j is only defined up to a sign) functions f and g such that

$$X = \frac{f(C_j)\phi_i - f(C_i)\phi_j}{g(C_j)\phi_i - g(C_i)\phi_j} \tag{1}$$

for every choice of i and j among the eight distinct pairs of τ at distance 2. Equating the values of X for two pairs $\{i, j\}$ and $\{k, m\}$ leads to a highly overdetermined system of equations for the τ. It turns out, however, that one can prove that this system will be compatible if we choose $f(x) \equiv \theta_1^2(\kappa x|m)$, $g(x) \equiv \theta_0^2(\kappa x|m)$ or a degenerate subcase thereof ($m \to 0$).

For this choice of f and g the equations obtained by writing the equality of X for two pairs sharing one element, say $\{i, j\}$ and $\{i, k\}$, involve only three products of two τ at distance 2 sharing the same midpoint. The equation assumes the form

$$\begin{aligned}(f(C_j)g(C_k) - f(C_k)g(C_j))\phi_i + (f(C_k)g(C_i) - f(C_i)g(C_k))\phi_j \\ + (f(C_i)g(C_j) - f(C_j)g(C_i))\phi_k = 0.\end{aligned} \tag{2}$$

These are nonautonomous Hirota–Miwa equations, overdetermined but guaranteed to be compatible by the choice of f and g.

Next if we consider two more points similar to X, say Y and Z, forming with X an equilateral triangle of side $1/\sqrt{2}$, one can check that there exist exactly six τ such that each of the points X, Y and Z can be written in terms of two pairs of points at distance 2, involving four of these six τ:

$$X = \frac{f(C_{X'})\phi_X - f(C_X)\phi_{X'}}{g(C_{X'})\phi_X - g(C_X)\phi_{X'}} \tag{3}$$

and similarly for Y and Z, with ϕ_X, etc, such that $\phi_X\phi_Y\phi_Z = \phi_{X'}\phi_{Y'}\phi_{Z'}$ is just the product of the six aforementioned τ. Solving (3) for $\phi_X/\phi_{X'}$ we find

$$\frac{\phi_X}{\phi_{X'}} = \frac{g(C_X)X - f(C_X)}{g(C_{X'})X - f(C_{X'})} \tag{4}$$

and taking the product we obtain the Miura relating X, Y and Z:

$$\frac{g(C_X)X - f(C_X)}{g(C_{X'})X - f(C_{X'})}\frac{g(C_Y)Y - f(C_Y)}{g(C_{Y'})Y - f(C_{Y'})}\frac{g(C_Z)Z - f(C_Z)}{g(C_{Z'})Z - f(C_{Z'})} = \frac{\phi_X}{\phi_{X'}}\frac{\phi_Y}{\phi_{Y'}}\frac{\phi_Z}{\phi_{Z'}} \equiv 1 \tag{5}$$

or using the function $h = f/g$, which is in general a Jacobi function $h = \mathrm{sn}^2(\kappa x; m)$ for elliptic-type equations but degenerates (for $m \to 0$) to $h = \sinh^2(\kappa x)$ for q-type equations or to $h = (\kappa x)^2$ for difference-type equations:

$$\frac{X - h(C_X)}{X - h(C_{X'})}\frac{Y - h(C_Y)}{Y - h(C_{Y'})}\frac{Z - h(C_Z)}{Z - h(C_{Z'})} = K \tag{6}$$

where K is some complicated expression in terms of Jacobi functions in the generic case and just unity in the q- and difference-cases.

Usually, one uses the Miura to derive an evolution equation, eliminating all intermediate variables, and presents the evolution on a straight line for one of the variables. Using this method, we obtained in [2] the explicit equations for the difference- and q-equation cases. Though, in principle, the same method could have been applied to the elliptic case, because of the prohibitative length of the calculations, we were not able, in practice, to exhibit any elliptic d-$\mathbb{P}$. This will be remedied in the present paper.

The alternative approach (which has been used, often implicitly, in previous works [5]) is to consider the variables entering the Miura as defining the variables in evolution (but in this case the trajectory of the evolution in the lattice is much more complicated). Based on the form of the Miura (6) we will introduce the mapping

$$\frac{x_{n+1}-c}{x_{n+1}-d}\frac{x_n-a}{x_n-b}\frac{x_{n-1}-p}{x_{n-1}-r}=s \tag{7}$$

where $a, b, \ldots, s$ are, in principle, functions of n. At first we limit ourselves to the autonomous limit and since (7) is a second-order mapping for a single independent variable we demand that it be of QRT form, i.e. part of the parametrization introduced in [6]. The computations are quite straightforward and it turns out that the condition is just $p = c, r = d$.

When these conditions are satisfied, it is possible to reduce the $\frac{x_{n+1}-c}{x_{n+1}-d}$, $\frac{x_{n-1}-p}{x_{n-1}-r}$ terms, through homographic transformations, to just x_{n+1} and x_{n-1}. In this case, the mapping is reduced to

$$x_{n+1}x_{n-1}=\gamma\frac{x_n-\alpha}{x_n-\beta} \tag{8}$$

i.e. a mapping already identified in [7]. The two well-known singularity patterns of mapping (8) are

(i) $\{\alpha, 0, \gamma/\beta, \infty, \beta\}$ and $\{\beta, \infty, \gamma/\beta, 0, \alpha\}$
(ii) $\{\beta, \infty, \beta\}$ and $\{\alpha, 0, \gamma/\beta, \infty, \gamma/\beta, 0, \alpha\}$

The first couple of singularity patterns corresponds to the generic case, where no special relation exits between the parameters of (8). The second couple exists only in the case $\gamma = \beta^2$. (A third case exists if $\gamma = \alpha\beta$, the short pattern being $\{\alpha, 0, \alpha\}$ but it is just obtained from the second one by inverting x, so it need not be considered separately.)

Next we proceed to deautonomize (8) using the singularity confinement criterion [8] (a procedure perfectly legitimate as we have explained in [9]). If we demand that the autonomous transformation for the reduction to the form (8) be valid also in the nonautonomous case, this means that we must have $c(n-1) = p(n+1) \equiv u(n)$ and $d(n-1) = r(n+1) \equiv v(n)$ and the homographic transformation must be $(x-u)/(x-v) \to x$. The deautonomization of mapping (8) based on the two singularity patterns above was first obtained in [7]. We start by gauging α to $1/\beta$ and obtain for the generic singularity pattern $\beta_{3n+k} = q^{3^n}\beta_k$ for $k = 0, 1, 2$ and $\gamma_n = g_{e,o}\beta_n$ where $g_{e,o}$ is an even–odd index-depending constant. For the nongeneric singularity pattern we obtain similarly $\beta_{5n+k} = q^{5^n}\beta_k$ for $k = 0, 1, 2, 3, 4$ and $\gamma_n = \beta_{n+1}\beta_{n-1}$. In both cases this equation is a new, unusual form of q-P_{VI} and the geometry of its transformations is described by the affine Weyl group $D_5^{(1)}$ [10].

We shall now turn to a deautonomization of (7) by taking special care that the solution we obtain has the full $E_8^{(1)}$ freedom and does not reduce to the $D_5^{(1)}$ solution above. Going through the calculational details for the confinement conditions would be overwhelming and, also, only moderately interesting. Thus we prefer to give the results directly. We start with the generic singularity, i.e. two patterns of length 5. For the difference- and q-cases, the singularity confinement of (7) can be carried explicitly. We find that $s = 1$, and moreover that $a = A^2$ (resp. $\sinh^2 \lambda A$) and similarly for b, c, etc, with

$$\begin{aligned}
A(n) &= -2n - \phi(n-1) - \phi(n+1) + \omega(n)\\
B(n) &= -2n - \phi(n-1) - \phi(n+1) - \omega(n)\\
C(n) &= n - \tfrac{1}{2} + \phi(n+1) + (-1)^n\psi + \omega(n)\\
D(n) &= n - \tfrac{1}{2} + \phi(n+1) + (-1)^n\psi - \omega(n)\\
P(n) &= n + \tfrac{1}{2} + \phi(n-1) - (-1)^n\psi + \omega(n)\\
R(n) &= n + \tfrac{1}{2} + \phi(n-1) - (-1)^n\psi - \omega(n)
\end{aligned} \tag{9}$$

where ψ is a constant, $\phi(n+3)=\phi(n)$, i.e. ϕ has a period 3 and $\omega(n+4)=\omega(n)$, i.e. ω has a period 4, so the whole equation has period 12. The total number of degrees of freedom is 8, including the independent variable. The explicit forms of the mappings are thus

$$\frac{x_{n-1}-\left(n+\frac{1}{2}+\phi(n-1)-(-1)^n\psi+\omega(n)\right)^2}{x_{n-1}-\left(n+\frac{1}{2}+\phi(n-1)-(-1)^n\psi-\omega(n)\right)^2}\,\frac{x_n-(-2n-\phi(n-1)-\phi(n+1)+\omega(n))^2}{x_n-(-2n-\phi(n-1)-\phi(n+1)-\omega(n))^2}$$
$$\times\frac{x_{n+1}-\lambda\left(n-\frac{1}{2}+\phi(n-1)+(-1)^n\psi+\omega(n)\right)^2}{x_{n+1}-\lambda\left(n-\frac{1}{2}+\phi(n-1)-(-1)^n\psi-\omega(n)\right)^2}=1 \tag{10}$$

$$\frac{x_{n-1}-\sinh^2\left(\lambda\left(n+\frac{1}{2}+\phi(n-1)-(-1)^n\psi+\omega(n)\right)\right)}{x_{n-1}-\sinh^2\left(\lambda\left(n+\frac{1}{2}+\phi(n-1)-(-1)^n\psi-\omega(n)\right)\right)}$$
$$\times\frac{x_n-\sinh^2(\lambda(-2n-\phi(n-1)-\phi(n+1)+\omega(n)))}{x_n-\sinh^2(\lambda(-2n-\phi(n-1)-\phi(n+1)-\omega(n)))}$$
$$\times\frac{x_{n+1}-\sinh^2\left(\lambda\left(n-\frac{1}{2}+\phi(n-1)+(-1)^n\psi+\omega(n)\right)\right)}{x_{n+1}-\sinh^2\left(\lambda\left(n-\frac{1}{2}+\phi(n-1)-(-1)^n\psi-\omega(n)\right)\right)}=1 \tag{11}$$

in the difference- and q-cases, respectively.

In the elliptic case, the complete analysis based on just the singularity confinement is extremely cumbersome. Fortunately, we were guided by the results we obtained in [2] from the geometry of $E_8^{(1)}$. We know from these results that exactly the same values for A, B, etc, in equation (9) will appear as the arguments of sn^2 in the elliptic discrete Painlevé equation. However, contrary to the difference- and q-cases the value of the rhs s is no longer unity. Its value can be computed from equation (6). We have finally

$$\frac{x_{n-1}-\mathrm{sn}^2\left(\lambda n+\frac{\lambda}{2}+\phi(n-1)-(-1)^n\psi+\omega(n);m\right)}{x_{n-1}-\mathrm{sn}^2\left(\lambda n+\frac{\lambda}{2}+\phi(n-1)-(-1)^n\psi-\omega(n);m\right)}$$
$$\times\frac{x_n-\mathrm{sn}^2(-2\lambda n-\phi(n-1)-\phi(n+1)+\omega(n);m)}{x_n-\mathrm{sn}^2(-2\lambda n-\phi(n-1)-\phi(n+1)-\omega(n);m)}$$
$$\times\frac{x_{n+1}-\mathrm{sn}^2\left(\lambda n-\frac{\lambda}{2}+\phi(n-1)+(-1)^n\psi+\omega(n);m\right)}{x_{n+1}-\mathrm{sn}^2\left(\lambda n-\frac{\lambda}{2}+\phi(n-1)-(-1)^n\psi-\omega(n);m\right)}$$
$$=\left[1-m^2\mathrm{sn}^2\left(\lambda n+\frac{\lambda}{6}+\frac{2\phi(n-1)+\phi(n+1)-(-1)^n\psi}{3}+\frac{\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(-\lambda n+\frac{\lambda}{6}-\frac{2\phi(n+1)+\phi(n-1)+(-1)^n\psi}{3}-\frac{\omega(n)}{2};m\right)\right]$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\lambda n+\frac{\lambda}{6}+\frac{2\phi(n-1)+\phi(n+1)-(-1)^n\psi}{3}-\frac{\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(-\lambda n+\frac{\lambda}{6}-\frac{2\phi(n+1)+\phi(n-1)+(-1)^n\psi}{3}+\frac{\omega(n)}{2};m\right)\right]^{-1}$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\lambda n+\frac{\lambda}{6}+\frac{2\phi(n-1)+\phi(n+1)-(-1)^n\psi}{3}-\frac{\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\phi(n+1)-\phi(n-1)+2(-1)^n\psi-\lambda}{3}+\frac{\omega(n)}{2};m\right)\right]$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\lambda n+\frac{\lambda}{6}+\frac{2\phi(n-1)+\phi(n+1)-(-1)^n\psi}{3}+\frac{\omega(n)}{2};m\right)\right.$$

$$\times\,\mathrm{sn}^2\left(\frac{\phi(n+1)-\phi(n-1)+2(-1)^n\psi-\lambda}{3}-\frac{\omega(n)}{2};m\right)\Bigg]^{-1}$$
$$\times\left[1-m^2\mathrm{sn}^2\left(-\lambda n+\frac{\lambda}{6}-\frac{2\phi(n+1)+\phi(n-1)+(-1)^n\psi}{3}+\frac{\omega(n)}{2};m\right)\right.$$
$$\times\,\mathrm{sn}^2\left(\frac{\phi(n+1)-\phi(n-1)+2(-1)^n\psi-\lambda}{3}-\frac{\omega(n)}{2};m\right)\Bigg]$$
$$\times\left[1-m^2\mathrm{sn}^2\left(-\lambda n+\frac{\lambda}{6}-\frac{2\phi(n+1)+\phi(n-1)+(-1)^n\psi}{3}-\frac{\omega(n)}{2};m\right)\right.$$
$$\times\,\mathrm{sn}^2\left(\frac{\phi(n+1)-\phi(n-1)+2(-1)^n\psi-\lambda}{3}+\frac{\omega(n)}{2};m\right)\Bigg]^{-1}. \tag{12}$$

Here in order to prevent the expressions getting even more lengthy, we have absorbed the scaling factor λ into the free objects ϕ, ψ and ω.

The nongeneric case, corresponding to the singularity patterns of lengths 3 and 7, can be treated in a completely similar way. The expressions of the parameters of (8) in terms of the capitalized ones are the same, but now

$$\begin{aligned}
A(n)&=-n+\phi(n)-\phi(n-2)-\phi(n+2)+\omega(n)\\
B(n)&=-3n-\phi(n)+\phi(n-2)-\phi(n+2)-\omega(n)\\
C(n)&=2n-\tfrac{1}{2}+\phi(n)+\phi(n+2)+(-1)^n\psi+\omega(n)\\
D(n)&=-\tfrac{1}{2}-\phi(n)+\phi(n+2)+(-1)^n\psi-\omega(n)\\
P(n)&=2n+\tfrac{1}{2}+\phi(n)+\phi(n-2)-(-1)^n\psi+\omega(n)\\
R(n)&=\tfrac{1}{2}-\phi(n)+\phi(n-2)-(-1)^n\psi-\omega(n)
\end{aligned} \tag{13}$$

where ψ is a constant, $\phi(n+5)=\phi(n)$, i.e. ϕ has period 5, and $\omega(n+2)=-\omega(n)$, so the period of ω is 4 but ω involves only two free parameters. The total periodicity of (7) is thus 20 (but the number of degree of freedom is again 8).

In this case we have $D(n-1)=-P(n+1)$ and, because the expressions of d and p are always even in D and P, $d(n-1)=p(n+1)$. But we do not have $c(n-1)=r(n+1)$, so we are not in a case reducible to $D_5^{(1)}$. The explicit forms of the difference- and q-mappings are

$$\frac{x_{n-1}-\left(2n+\frac{1}{2}+\phi(n)+\phi(n-2)-(-1)^n\psi+\omega(n)\right)^2}{x_{n-1}-\left(\frac{1}{2}-\phi(n)+\phi(n-2)-(-1)^n\psi-\omega(n)\right)^2}
\times\frac{x_n-(-n+\phi(n)-\phi(n-2)-\phi(n+2)+\omega(n))^2}{x_n-(-3n-\phi(n)+\phi(n-2)-\phi(n+2)-\omega(n))^2}
\times\frac{x_{n+1}-\left(2n-\frac{1}{2}+\phi(n)+\phi(n+2)+(-1)^n\psi+\omega(n)\right)^2}{x_{n+1}-\left(-\frac{1}{2}-\phi(n)+\phi(n+2)+(-1)^n\psi-\omega(n)\right)^2}=1 \tag{14}$$

$$\frac{x_{n-1}-\sinh^2\left(\lambda\left(2n+\frac{1}{2}+\phi(n)+\phi(n-2)-(-1)^n\psi+\omega(n)\right)\right)}{x_{n-1}-\sinh^2\left(\lambda\left(\frac{1}{2}-\phi(n)+\phi(n-2)-(-1)^n\psi-\omega(n)\right)\right)}
\times\frac{x_n-\sinh^2(\lambda(-n+\phi(n)-\phi(n-2)-\phi(n+2)+\omega(n)))}{x_n-\sinh^2(\lambda(-3n-\phi(n)+\phi(n-2)-\phi(n+2)-\omega(n)))}
\times\frac{x_{n+1}-\sinh^2\left(\lambda\left(2n-\frac{1}{2}+\phi(n)+\phi(n+2)+(-1)^n\psi+\omega(n)\right)\right)}{x_{n+1}-\sinh^2\left(\lambda\left(-\frac{1}{2}-\phi(n)+\phi(n+2)+(-1)^n\psi-\omega(n)\right)\right)}=1. \tag{15}$$

In the elliptic case, what we said above also applies here. The only difference is the expression for s:

$$\frac{x_{n-1}-\mathrm{sn}^2\big(2\lambda n+\frac{\lambda}{2}+\phi(n)+\phi(n-2)-(-1)^n\psi+\omega(n);m\big)}{x_{n-1}-\mathrm{sn}^2\big(\frac{\lambda}{2}-\phi(n)+\phi(n-2)-(-1)^n\psi-\omega(n);m\big)}$$
$$\times\frac{x_n-\mathrm{sn}^2(-\lambda n+\phi(n)-\phi(n-2)-\phi(n+2)+\omega(n);m)}{x_n-\mathrm{sn}^2(-3\lambda n-\phi(n)+\phi(n-2)-\phi(n+2)-\omega(n);m)}$$
$$\times\frac{x_{n+1}-\mathrm{sn}^2\big(2\lambda n-\frac{\lambda}{2}+\phi(n)+\phi(n+2)+(-1)^n\psi+\omega(n);m\big)}{x_{n+1}-\mathrm{sn}^2\big(-\frac{\lambda}{2}-\phi(n)+\phi(n+2)+(-1)^n\psi-\omega(n);m\big)}$$
$$=\left[1-m^2\mathrm{sn}^2\left(\frac{\lambda}{6}+\frac{2\phi(n-2)+\phi(n+2)-(-1)^n\psi}{3}+\frac{3\lambda n+\phi(n)+\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\lambda}{6}-\frac{2\phi(n+2)+\phi(n-2)+(-1)^n\psi}{3}-\frac{3\lambda n+\phi(n)+\omega(n)}{2};m\right)\right]$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\frac{\lambda}{6}+\frac{2\phi(n-2)+\phi(n+2)-(-1)^n\psi}{3}+\frac{\lambda n-\phi(n)-\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\lambda}{6}-\frac{2\phi(n+2)+\phi(n-2)+(-1)^n\psi}{3}-\frac{\lambda n-\phi(n)-\omega(n)}{2};m\right)\right]^{-1}$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\frac{\lambda}{6}+\frac{2\phi(n-2)+\phi(n+2)-(-1)^n\psi}{3}+\frac{\lambda n-\phi(n)-\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\phi(n+2)-\phi(n-2)+2(-1)^n\psi-\lambda}{3}+\frac{\lambda n+\phi(n)+\omega(n)}{2};m\right)\right]$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\frac{\lambda}{6}+\frac{2\phi(n-2)+\phi(n+2)-(-1)^n\psi}{3}+\frac{3\lambda n+\phi(n)+\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\phi(n+2)-\phi(n-2)+2(-1)^n\psi-\lambda}{3}-\frac{\lambda n+\phi(n)+\omega(n)}{2};m\right)\right]^{-1}$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\frac{\lambda}{6}-\frac{2\phi(n+2)+\phi(n-2)+(-1)^n\psi}{3}-\frac{\lambda n-\phi(n)-\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\phi(n+2)-\phi(n-2)+2(-1)^n\psi-\lambda}{3}-\frac{\lambda n+\phi(n)+\omega(n)}{2};m\right)\right]$$
$$\times\left[1-m^2\mathrm{sn}^2\left(\frac{\lambda}{6}-\frac{2\phi(n+2)+\phi(n-2)+(-1)^n\psi}{3}-\frac{3\lambda n+\phi(n)+\omega(n)}{2};m\right)\right.$$
$$\left.\times\,\mathrm{sn}^2\left(\frac{\phi(n+2)-\phi(n-2)+2(-1)^n\psi-\lambda}{3}+\frac{\lambda n+\phi(n)+\omega(n)}{2};m\right)\right]^{-1}\qquad(16)$$

where we have similarly absorbed the scaling factor λ into the free objects ϕ ψ and ω.

Equations (12) and (16) constitute explicit examples of elliptic discrete Painlevé equations. Since they involve more than five degrees of freedom, these d-$\mathbb{P}$ go beyond the richness of $\mathrm{P_{VI}}$. It must be stressed here that these equations, although they look awesome, are probably the simplest ones we can write. As a matter of fact, in [2] we have presented the explicit forms of difference- and q-equations, the geometry of which is described by $E_8^{(1)}$. These equations were considerably more complicated than the corresponding forms obtained here. The complexity of these calculations was in fact what prevented us from obtaining elliptic discrete mappings. This led us to the present paper where elliptic forms of discrete Painlevé equations are explicitly obtained. More elliptic discrete Painlevé equations should exist. In fact any nonclosed periodically repeated pattern in $E_8^{(1)}$ would lead to such an equation.

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